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Minimal Polynomials of Additive Commutators and Jordan Products*

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1. INTRODUCTION

Let A be an n-square matrix over a field F of characteristic 0 and consider the linear transformations T_A and L_A defined on the space $M_n(F)$ of n-square matrices over F:

$$T_A(X) = AX - XA,$$
$$L_A(X) = \tfrac{1}{2}(AX + XA), \qquad X \in M_n(F).$$

Then T_A is called the *commutator operator* and L_A the *Jordan operator* defined by A. Let $\gamma_1, \ldots, \gamma_k$ be the distinct eigenvalues of A in an appropriate algebraic extension field K of F. Then the elementary divisors over K of the characteristic matrix $\lambda I_n - A$ of A are powers of the binomials $\lambda - \gamma_i$. Let e_i denote the degree of the highest degree elementary divisor of $\lambda I_n - A$ involving γ_i, $i = 1, \ldots, k$, and let $E = \max_i e_i$, $e = \min_i e_i$ and m be the degree of the minimal polynomial of A. Let S_n denote the symmetric group of degree n and let Q be the field of rational numbers.

Annihilating polynomials for commutators were first considered by Taussky and Wielandt in a paper in 1962 [6]. In 1964, one of the present authors determined an upper bound on the degree of the minimal polynomial of T_A [2] (see also [1] and [7]). In a recent paper [3], the authors proved the following result:

THEOREM 1.1. *If d_A is the degree of the minimal polynomial of the commutator operator T_A then*

(i) *d_A is always odd, and*
(ii) $d_A \geqslant 2(m + E + (k - 2)e - k) + 1.$ (1)

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The purpose of this paper is to prove the following results:

THEOREM 1.2. *If $k \leqslant 2$, then equality holds in* (1). *If $k \geqslant 3$ then the equality holds in* (1) *if and only if there exist elements a and b in K, $b \neq 0$ and a permutation φ in S_k such that*

$$\text{(i)} \quad \gamma_{\varphi(t)} = a + tb, \qquad t = 1,..., k \tag{2}$$

and

$$\begin{aligned} \text{(ii)} \quad e_{\varphi(1)} &= e_{\varphi(3)} \leqslant e_{\varphi(2)} && \textit{if} \quad k = 3, \\ e_1 = e_2 &= \cdots = e_k && \textit{if} \quad k > 3. \end{aligned} \tag{3}$$

In order to prove Theorem 1.2 we shall find it necessary to prove the following statement:

THEOREM 1.3. *Let $z_1, ..., z_k$ be k distinct elements in a field F of characteristic 0. Then the number of nonzero distinct differences of the form $z_i - z_j$, $i \neq j$, $i, j = 1,..., k$, is at least $2k - 2$. It is exactly $2k - 2$ if and only if there exist elements $a, b \in F$, $b \neq 0$ and a permutation $\varphi \in S_k$ such that*

$$z_{\varphi(t)} = a + tb, \qquad t = 1,..., k. \tag{4}$$

It may be remarked here that the first part of the above theorem has already been proved by the authors in [3]. However, we will give a different proof in this paper that makes the case of equality accessible.

Next we shall determine another lower bound for d_A which is sometimes better than the one given in Theorem 1.1 (see Section 3 for examples).

THEOREM 1.4. *If d_A is the degree of the minimal polynomial of T_A, then*

$$d_A \geqslant kE + m + 2(k - 1)e - 2k + 1. \tag{5}$$

Moreover, if k is even, then

$$d_A \geqslant (k + 1)E + m + (2k - 3)e - 2k + 1. \tag{6}$$

We remark that if k is even, then the lower bound in (6) is always greater than or equal to any of the bounds in (5) and (1).

Our final result about the minimal polynomial of T_A is contained in

THEOREM 1.5. *If d_A is 1, 3 or 5 then A has at least one of the following two properties*:

(i) *A is similar to a diagonal matrix with $\frac{1}{2}(d_A + 1)$ distinct eigenvalues.*

(ii) *There exists an element* $\alpha \in K$ *such that* $A - \alpha I_n$ *is nilpotent of index* $\frac{1}{2}(d_A + 1)$.

Moreover, for each odd number $p \geqslant 7$ *there exists a matrix* $A \in M_n(F)$ *such that* $d_A = p$ *but* A *does not satisfy either of the conditions* (i) *and* (ii).

We shall also obtain a lower bound for the degree of the minimal polynomial of the Jordan operator L_A and prove a result somewhat analogous to Theorem 1.5.

THEOREM 1.6. *Let* F *be a field of characteristic* 0. *For* $A \in M_n(F)$ *let* δ_A *denote the degree of the minimal polynomial of* L_A. *Then*

$$\delta_A \geqslant 2(2m - k - E) + 1. \tag{7}$$

Moreover, equality holds in (7) *if and only if there exist elements* a *and* b *in* K, $b \neq 0$ *and a permutation* $\varphi \in S_k$ *such that*

(i) $\gamma_{\varphi(t)} = a + bt, \qquad t = 1,..., k,$

and

(ii) $e_1 = e_2 = \cdots = e_k$.

In order to prove Theorem 1.6 we need to consider the following problem: Given k distinct numbers $z_1, ..., z_k$ in F how many distinct sums are there of the form $z_i + z_j$, $i, j = 1,..., k$? The answer to this is contained in the following result:

THEOREM 1.7. *Let* $z_1, ..., z_k$ *be* k *distinct elements in a field* F *of characteristic* 0. *Let* $\mathscr{S}$ *be the set of distinct sums of the form* $z_i + z_j$, $i, j = 1,..., k$ *and* $\nu(\mathscr{S})$ *denote the cardinality of* $\mathscr{S}$. *Then*

(a) *There is a permutation* $\varphi \in S_k$ *such that the set*

$$M = \{2z_i : i = 1,..., k\} \cup \{z_{\varphi(i)} + z_{\varphi(i+1)} : i = 1,..., k - 1\}$$

consists of distinct elements and hence $\nu(\mathscr{S}) \geqslant 2k - 1$.

(b) $\nu(\mathscr{S}) = 2k - 1$ *if and only if there exist a permutation* $\varphi \in S_k$ *and elements* $a, b \in F$, $b \neq 0$, *such that*

$$z_{\varphi(t)} = a + bt, \qquad t = 1,..., k. \tag{8}$$

To a certain extent, our final result answers the following question about the Jordan product: Given a positive integer p, what can be said about A if $\delta_A = p$?

THEOREM 1.8. (i) *There is no matrix* $A \in M_n(F)$ *such that* $\delta_A = 2$ *or* 4.

(ii) *If* δ_A *is* 1, 3, 5, 7 *or* 11, *then* A *satisfies at least one of the following two conditions*:

(a) A *is similar to a diagonal matrix with* $\frac{1}{2}(\delta_A + 1)$ *distinct eigenvalues if* $\delta_A = 1, 3, 5, 7$ *and* $\frac{1}{2}(\delta_A \pm 1)$ *distinct eigenvalues if* $\delta_A = 11$.

(b) *There exists an element* $\alpha \in K$ *such that* $A - \alpha I_n$ *is nilpotent of index* $\frac{1}{2}(\delta_A + 1)$.

(iii) *If* $\delta_A = 8$ *then* A *is similar to a diagonal matrix with four distinct eigenvalues.*

(iv) *For each of the positive integers* $p = 6, 9, 10, 12, 13, 14, \ldots$, *there exists a matrix* $A \in M_n(F)$ *such that* $\delta_A = p$ *but* A *does not satisfy either of the conditions* (a) *and* (b) *in* (ii).

2. PROOFS

Let V be an n-dimensional vector space over a field F of characteristic 0 and let V^* be the dual space of V. Let $g_1, \ldots, g_n$ be the basis of V^* dual to a given basis $e_1, \ldots, e_n$ of V. Let $u_1, \ldots, u_p$ be distinct vectors of V and let $Q_{2,p}$ denote the set of all pairs $\alpha = (\alpha(1), \alpha(2))$ such that $\alpha(1) < \alpha(2)$, $\alpha(1), \alpha(2) = 1, \ldots, p$. Then for each $\alpha \in Q_{2,p}$ there are scalars $c_{\alpha t}$, $t = 1, \ldots, n$ such that

$$u_{\alpha(1)} - u_{\alpha(2)} = \sum_{t=1}^{n} c_{\alpha t} e_t .$$

Let Y be an n-square matrix of n^2 independent indeterminates x_{ij}, $i, j = 1, \ldots, n$ over F and let $X = Y^{\top}$. Then $\det(X) = \det(Y)$ is a nonzero polynomial. Set $f_s = \sum_{t=1}^{n} x_{st} g_t$, $s = 1, \ldots, n$ and define a polynomial

$$P(x_{11}, x_{12}, \ldots, x_{nn}) = \det(Y) \prod_{s=1}^{n} \prod_{\alpha \in Q_{2,p}} f_s(u_{\alpha(1)} - u_{\alpha(2)}).$$

Now

$$\begin{aligned} f_s(u_{\alpha(1)} - u_{\alpha(2)}) &= \sum_{t=1}^{n} c_{\alpha t} f_s(e_t) \\ &= \sum_{t=1}^{n} c_{\alpha t} \sum_{k=1}^{n} x_{sk} g_k(e_t) \\ &= \sum_{t=1}^{n} c_{\alpha t} x_{st} = \sum_{t=1}^{n} c_{\alpha t} y_{ts} . \end{aligned}$$

Hence if $P(x_{11}, \ldots, x_{nn})$ is identically 0, then there is an $\alpha \in Q_{2,p}$ and

$s \in \{1,..., n\}$ such that $f_s(u_{\alpha(1)} - u_{\alpha(2)}) = \sum_{t=1}^{n} c_{\alpha t}\, y_{ts} = 0$. But this implies that $c_{\alpha t} = 0$, $t = 1,..., n$, i.e., $u_{\alpha(1)} - u_{\alpha(2)} = 0$. This contradicts the assumption that the u_i's are distinct. Thus the polynomial $P(x_{11},..., x_{nn})$ is not identically 0 and since the characteristic of F is 0 there exists a specialization b_{ij} of x_{ij}, $i, j = 1,..., n$ such that $P(b_{11},..., b_{nn}) \neq 0$. Hence there exists a matrix $A = [b_{ij}]^{\mathsf{T}} \in M_n(F)$ such that if

$$f_s = \sum_{k=1}^{n} a_{sk} g_k\,, \qquad s = 1,..., n,$$

then

$$\det(A) \prod_{s=1}^{n} \prod_{\alpha \in Q_{2,p}} f_s(u_{\alpha(1)} - u_{\alpha(2)}) \neq 0.$$

It follows that $f_s(u_{\alpha(1)}) \neq f_s(u_{\alpha(2)})$, $s = 1,..., n$, $\alpha \in Q_{2,p}$ and that $f_1,..., f_n$ are linearly independent. Now let $e_1,..., e_n$ be a basis of V with respect to which $f_1,..., f_n$ is dual. Write $u_i = \sum_{j=1}^{n} c_{ij} e_j$, and observe that

$$f_s(u_i) = \sum_{j=1}^{n} c_{ij} f_s(e_j) = c_{is}\,.$$

Hence for each $s = 1,..., n$, the p numbers c_{is}, $i = 1,..., p$ are distinct. We have therefore proved the following lemma:

LEMMA 2.1. *Assume that $u_1,..., u_p$ are distinct elements of an n-dimensional vector space V over a field F of characteristic* 0. *Then*

(a) *There exist linearly independent $f_s \in V^*$, $s = 1,..., n$ such that $f_s(u_i)$, $i = 1,..., p$ are distinct elements of F for each $s = 1,..., n$.*

(b) *There exists a basis $e_1,..., e_n$ of V such that if $u_i = \sum_{j=1}^{n} c_{ij} e_j$, then the scalars $c_{1j},..., c_{pj}$ are distinct, $j = 1,..., n$. Hence the number of distinct sums $u_i + u_j$ (distinct differences $u_i - u_j$) $i, j = 1,..., p$ is greater than or equal to the number of distinct sums $c_{it} + c_{jt}$ (distinct differences $c_{it} - c_{jt}$), $i, j = 1,..., p$ for each $t = 1,..., n$.*

LEMMA 2.2. *Let $r_1,..., r_p$ be distinct rational numbers and let $\mathscr{D}$ be the totality of nonzero distinct differences of the form $r_i - r_j$, $i, j = 1,..., p$, $i \neq j$. Then*

(a) *There exists $\varphi \in S_p$ such that the set $\{\pm(r_{\varphi(i)} - r_{\varphi(1)} : i = 2,..., p\}$ consists of distinct elements and hence $\nu(\mathscr{D}) \geqslant 2p - 2$.*

(b) *$\nu(\mathscr{D}) = 2p - 2$ if and only if there exist $\varphi \in S_p$ and elements a and b in Q, $b \neq 0$ such that*

$$r_{\varphi(t)} = a + bt, \qquad t = 1,..., p. \tag{9}$$

Proof. (a) Let $\varphi \in S_p$ be such that $r_{\varphi(1)} < r_{\varphi(2)} < \cdots < r_{\varphi(p)}$ and consider the set $\{\pm(r_{\varphi(i)} - r_{\varphi(1)}) : i = 2,..., p\}$. The result follows.

(b) It is obvious that if (9) holds, then $\nu(\mathscr{D}) = 2p - 2$. To prove the converse assume that $\nu(\mathscr{D}) = 2p - 2$. Let $\varphi \in S_p$ be such that

$$r_{\varphi(1)} < r_{\varphi(2)} < \cdots < r_{\varphi(p)} .$$

Set $r_{\varphi(i)} = s_i$, $i = 1,..., p$. Since the number of nonzero distinct differences involving $s_1 ,..., s_p$ remains the same if each s_i is multiplied by a nonzero rational number or each s_i is translated by the same rational number, it suffices to show that

$$s_{i+1} - s_i = b, \qquad i = 1,..., p - 1$$

for some $b \in Q$, $b \neq 0$.

There is nothing to prove if $p = 2$. So assume $p \geqslant 3$. Since

$$\nu(\{\pm(s_i - s_1) : i = 2,..., p\}) = 2p - 2,$$

we conclude that

$$\mathscr{D} = \{\pm(s_i - s_1) : i = 2,..., p\}.$$

We use induction on p. If $p = 3$, then consider $s_3 - s_2$. Since $s_3 - s_2 \in \mathscr{D}$ and $s_3 - s_2 > 0$, we have two possibilities, namely, $s_3 - s_2 = s_2 - s_1 = b$ or $s_3 - s_2 = s_3 - s_1$. But the latter case contradicts the assumption that $s_2 \neq s_1$. Now assume the result to be true for all positive integers $n \leqslant p - 1$. Consider the set $\mathscr{B} = \{\pm(s_i - s_1) : i = 2,..., p - 1\}$. If there is some nonzero difference $s_i - s_j$, $i \neq j$, $i, j = 1,..., p - 1$, which is not in $\mathscr{B}$, then either (i) $s_i - s_j = s_p - s_1$ or (ii) $s_i - s_j = s_1 - s_p$. Assume without loss of generality that $i > j$. Then $s_i - s_j > 0$ and thus (ii) cannot hold. Also, (i) implies that $0 < s_p - s_i = s_1 - s_j < 0$. Hence $\mathscr{B}$ is the set of all nonzero distinct differences involving $s_1 ,..., s_{p-1}$ and $\nu(\mathscr{B}) = 2p - 4$. Therefore, by the induction hypothesis, $s_{i+1} - s_i = b$, $i = 1,..., p - 2$. We have

$$\begin{aligned} s_p - s_2 &= (s_p - s_{p-1}) + \sum_{i=3}^{p-1} s_i - s_{i-1} \\ &= (s_p - s_{p-1}) + (p - 3)b. \end{aligned} \tag{10}$$

Now $p \geqslant 3$, and so $s_p - s_2 > 0$. Hence

$$s_p - s_2 = s_j - s_1$$

for some $j \leqslant p - 1$. Then (10) gives

$$\begin{aligned} 0 < s_p - s_{p-1} &= s_j - s_1 - (p-3)b \\ &= (j-1)b - (p-3)b \\ &= (j+2-p)b. \end{aligned}$$

Hence $j > p - 2$. Thus $j = p - 1$ and $s_p - s_{p-1} = b$. This completes the proof of the lemma.

The following combinatorial result may be of some independent interest.

LEMMA 2.3. *Let $\varphi \in S_p$. Then the number of distinct vectors in $V_2(Q)$ of the form $(i - j, \varphi(i) - \varphi(j))$, $i, j = 1,..., p$, is exactly $2p - 1$ if and only if either φ is the identity permutation e or $\varphi(t) = p - t + 1$, $t = 1,..., p$.*

Proof. Construct the following scheme (see next page).

Consider the set of all pairs of differences in the first column together with the last pair of differences in each of the remaining columns. This set, namely,

$$\{(1-p, \varphi(1) - \varphi(p)), (2-p, \varphi(2) - \varphi(p)),..., (0, 0), (1, \varphi(p) - \varphi(p-1)), \\ (2, \varphi(p) - \varphi(p-2)),..., (p-1, \varphi(p) - \varphi(1))\}$$

contains $2p - 1$ distinct vectors. Thus the number of distinct vectors of the form $(i - j, \varphi(i) - \varphi(j))$, $i, j = 1,..., p$ is $2p - 1$ if and only if each row of the given scheme contains only one vector. This is true if and only if

$$\varphi(p) - \varphi(p-1) = \varphi(p-1) - \varphi(p-2) = \cdots = \varphi(2) - \varphi(1).$$

Let $\varphi(p) = k$ and $\varphi(p - 1) = k + l$. Then

$$-l = (k + l) - \varphi(p-2) = \varphi(p-2) - \varphi(p-3) = \cdots = \varphi(2) - \varphi(1),$$

i.e.,

$$\varphi(p - t) = k + tl, \qquad t = 0, 1,..., p - 1.$$

Now if $l > 0$, then $k + (p - 1)l = p$ and $k = 1$ which implies that $l = 1$ and $\varphi(t) = p - t + 1$, $t = 1,..., p$. On the other hand, if $l < 0$, then $k + (p - 1)l = 1$ and $k = p$. This gives $l = -1$ and $\varphi = e$.

Proof of Theorem 1.3. Let W be the set of all linear combinations of $z_1,..., z_k$ in which the coefficients come from the rational subfield of F. We shall also denote this rational subfield by Q. Then W is a vector space over Q. By Lemma 2.1(b) there exists a basis $e_1,..., e_n$ of W, $n \leqslant k$, such that if $z_i = \sum_{j=1}^{n} c_{ij} e_j$, $i = 1,..., k$ then each of the sets $\{c_{ij} : i = 1,..., k\}$, $j = 1,..., n$ consists of distinct rational numbers. If $\mathscr{D}$ denotes the number of

$i - p, \varphi(i) - \varphi(p)$	$i - (p - 1), \varphi(i) - \varphi(p - 1)$	$i - (p - 2), \varphi(i) - \varphi(p - 2)$	$\cdots$	$i - 1, \varphi(i) - \varphi(1)$
$1 - p, \varphi(1) - \varphi(p)$				
$2 - p, \varphi(2) - \varphi(p)$	$2 - p, \varphi(1) - \varphi(p - 1)$			
$3 - p, \varphi(3) - \varphi(p)$	$3 - p, \varphi(2) - \varphi(p - 1)$	$3 - p, \varphi(1) - \varphi(p - 2)$		
$4 - p, \varphi(4) - \varphi(p)$	$4 - p, \varphi(3) - \varphi(p - 1)$	$4 - p, \varphi(2) - \varphi(p - 2)$	$\cdots$	
.	.	.	$\cdots$	
.	.	.	$\cdots$	
.	.	.	$\cdots$	
$-1, \varphi(p - 1) - \varphi(p)$	$-1, \varphi(p - 2) - \varphi(p - 1)$	$-1, \varphi(p - 3) - \varphi(p - 2)$	$\cdots$	
$0, 0$	$0, 0$	$0, 0$	$\cdots$	$0, 0$
	$1, \varphi(p) - \varphi(p - 1)$	$1, \varphi(p - 1) - \varphi(p - 2)$	$\cdots$	$1, \varphi(2) - \varphi(1)$
		$2, \varphi(p) - \varphi(p - 2)$	$\cdots$	$2, \varphi(3) - \varphi(1)$
			$\cdots$	$3, \varphi(4) - \varphi(1)$
				.
				.
				.
				$p - 1, \varphi(p) - \varphi(1)$

nonzero distinct differences of the form $z_i - z_j$, $i \neq j$, $i, j = 1, \ldots, k$ then, by Lemma 2.1(b) and Lemma 2.2(a), there exists $\varphi \in S_k$ such that the differences $\pm(z_{\varphi(i)} - z_{\varphi(1)})$, $i = 2, \ldots, k$ are distinct and hence $\nu(\mathscr{D}) \geqslant 2k - 2$.

Next suppose that $\nu(\mathscr{D}) = 2k - 2$. Then, again by Lemma 2.1(b) the number of distinct nonzero differences involving the rational numbers $c_{1j}, c_{2j}, \ldots, c_{kj}$ is exactly $2k - 2$ for any $j = 1, \ldots, n$. By Lemma 2.2(b) there exist $\varphi_t \in S_k$, a_t, $b_t \in Q$, $b_t \neq 0$, $t = 1, \ldots, n$ such that $c_{it} = a_t + b_t\varphi_t(i)$, $i = 1, \ldots, k$. Then

$$z_i = \sum_{j=1}^{n} (a_j + b_j\varphi_j(i))\, e_j, \qquad i = 1, \ldots, k.$$

Replacing e_1 by $(1/b_1)(e_1)$ we can assume that $b_1 = 1$. By reordering the z_i's we can also assume that $\varphi_1 = e$. Now consider the elements

$$y_i = z_i - \sum_{j=1}^{n} a_j e_j = z_i - a, \qquad i = 1, \ldots, k.$$

Then the number of distinct nonzero differences involving the y_i's is $\nu(\mathscr{D}) = 2k - 2$. Thus we can assume $a_i = 0$, $i = 1, \ldots, k$, and write

$$z_i = ie_1 + \sum_{j=2}^{n} b_j\varphi_j(i)\, e_j, \qquad i = 1, \ldots, k.$$

If $n = 1$, i.e., dim $W = 1$, then we are finished. Therefore, assume $n > 1$. Hence for $i \neq j$ and $t > 1$ we have

$$z_i - z_j = (i - j)e_1 + (\varphi_t(i) - \varphi_t(j))\, b_t e_t + l_t,$$

where l_t is a linear combination of e_s for $s \neq 1$, $s \neq t$. There are $2k - 2$ distinct nonzero differences of the form $z_i - z_j$, $i \neq j$, $i, j = 1, \ldots, k$. Thus there are $2k - 2$ distinct nonzero pairs of the form $(i - j, \varphi_t(i) - \varphi_t(j))$. Therefore, by Lemma 2.3, $\varphi_t = \varphi_0$ or $\varphi_t = e$, $t = 2, \ldots, k$, where

$$\varphi_0 = \begin{pmatrix} 1 & 2 & \cdots & k \\ k & k-1 & \cdots & 1 \end{pmatrix}.$$

Suppose $\varphi_{t_1} = \cdots = \varphi_{t_p} = e$ and $\varphi_{s_1} = \cdots = \varphi_{s_q} = \varphi_0$, $p + q = n$. Then

$$\begin{aligned} z_i &= i \sum_{j=1}^{p} b_{t_j} e_{t_j} + \varphi_0(i) \sum_{j=1}^{q} b_{s_j} e_{s_j} \\ &= a_1 i + (k - i + 1)\, d_1 \\ &= (k + 1)\, d_1 + (a_1 - d_1)i \\ &= a + di, \qquad i = 1, \ldots, k. \end{aligned}$$

The proof is complete.

In order to prove Theorem 1.2 we first briefly recapitulate the proof of Theorem 1.1(ii) as given in [3]. The commutator operator T_A has a matrix representation $A \otimes I_n - I_n \otimes A$, where $\otimes$ denotes the Kronecker product [4]. Corresponding to the elementary divisors $(\lambda - \gamma_i)^p$ and $(\lambda - \gamma_j)^q$ of $\lambda I_n - A$ the elementary divisors of the characteristic matrix of $A \otimes I_n - I_n \otimes A$ involving the eigenvalue $\gamma_i - \gamma_j$ are

$$(\lambda - (\gamma_i - \gamma_j))^{p+q-(2t-1)}, \qquad t = 1,\ldots, \min\{p, q\} \quad [5].$$

It has been proved in [3] that the number of nonzero distinct differences $\gamma_i - \gamma_j$, $i \neq j$, $i, j = 1,\ldots, k$ is always even. Let $\varphi \in S_k$ be such that the differences $\pm(\gamma_{\varphi(1)} - \gamma_{\varphi(i)})$, $i = 2,\ldots, k$ are all distinct (see the proof of Theorem 1.3). So suppose that the totality of nonzero distinct differences is

$$\mathscr{D} = \{\pm(\gamma_{\varphi(1)} - \gamma_{\varphi(i)}) : i = 2,\ldots, k\} \cup \{\pm(\gamma_{i_t} - \gamma_{j_t}) : t = 1,\ldots, p\}.$$

The highest degree elementary divisor of the characteristic matrix of $A \otimes I_n - I_n \otimes A$ involving the zero eigenvalue is λ^{2E-1}. Assume that

$$(\lambda \pm (\gamma_{\varphi(1)} - \gamma_{\varphi(i)}))^{e_{r_i}+e_{s_i}-1}$$

are the highest degree elementary divisors corresponding to the eigenvalues $\pm(\gamma_{\varphi(i)} - \gamma_{\varphi(1)})$, $i = 2,\ldots, k$, and

$$(\lambda \pm (\gamma_{i_t} - \gamma_{j_t}))^{e_{m_t}+e_{q_t}-1}$$

are the highest degree elementary divisors corresponding to the eigenvalues $\pm(\gamma_{j_t} - \gamma_{i_t})$, $t = 1,\ldots, p$. Then since the minimal polynomial of any linear transformation is the product of all the distinct highest degree elementary divisors we have

$$\begin{aligned} d_A &= 2E - 1 + 2\sum_{i=2}^{k} (e_{r_i} + e_{s_i} - 1) + 2\sum_{t=1}^{p} (e_{m_t} + e_{q_t} - 1) \\ &\geqslant 2E - 1 + 2\sum_{i=2}^{k} (e_{\varphi(1)} + e_{\varphi(i)} - 1) + 2\sum_{t=1}^{p} (e_{i_t} + e_{j_t} - 1) \qquad (11) \\ &\geqslant 2E - 1 + 2\sum_{i=2}^{k} (e_{\varphi(1)} + e_{\varphi(i)} - 1) \qquad (12) \\ &= 2(E + m + (k-2)\, e_{\varphi(1)} - k) + 1 \\ &\geqslant 2(m + E + (k-2)e - k) + 1. \qquad (13) \end{aligned}$$

Proof of Theorem 1.2. If k is 1 or 2, then the number of distinct eigenvalues of T_A is 1 or 3, respectively, and it is easily verified that in this case equality always holds in (1). For $k \geqslant 3$ the equality holds in (1) if and only if it holds in (11)–(13). Now equality holds in (11) if and only if

$$e_{\varphi(1)} + e_{\varphi(i)} \geqslant e_{h_1} + e_{h_2}, \tag{14}$$

whenever $\gamma_{\varphi(1)} - \gamma_{\varphi(i)} = \gamma_{h_1} - \gamma_{h_2}$, and

$$e_{i_t} + e_{j_t} \geqslant e_{l_1} + e_{l_2},$$

whenever $\gamma_{i_t} - \gamma_{j_t} = \gamma_{l_1} - \gamma_{l_2}$. The equality holds in (12) if and only if the number of distinct nonzero eigenvalues of $A \otimes I_n - I_n \otimes A$ is $2k - 2$ which, by Theorem 1.3, is true if and only if there exist $\theta \in S_k$, $a, b \in K$, $b \neq 0$ such that

$$\gamma_{\theta(t)} = a + tb, \qquad t = 1,\ldots, k. \tag{15}$$

The only condition on the permutation φ is that the $2k - 2$ differences $\pm(\gamma_{\varphi(1)} - \gamma_{\varphi(i)})$, $i = 2,\ldots, k$ are distinct. Since this is clearly true for the permutation θ satisfying (15) we may take $\varphi = \theta$. Finally, equality holds in (13) if and only if

$$e_{\varphi(1)} = e. \tag{16}$$

From (15) we have

$$\gamma_{\varphi(2)} - \gamma_{\varphi(1)} = b = \gamma_{\varphi(3)} - \gamma_{\varphi(2)},$$

and it follows from (14) that

$$e_{\varphi(2)} + e_{\varphi(1)} \geqslant e_{\varphi(3)} + e_{\varphi(2)}.$$

Then using (16), we get

$$e_{\varphi(1)} = e_{\varphi(3)} = e.$$

Thus if $k = 3$, then $e_{\varphi(1)} = e_{\varphi(3)} \leqslant e_{\varphi(2)}$. If $k > 3$, then we assert that conditions (14)–(16) are equivalent to the condition (15) and the equalities

$$e_1 = e_2 = \cdots = e_k. \tag{17}$$

For, as above, $e_{\varphi(1)} = e_{\varphi(3)} = e$. Now, since $\gamma_{\varphi(3)} - \gamma_{\varphi(1)} = 2b = \gamma_{\varphi(4)} - \gamma_{\varphi(2)}$, we have

$$\begin{aligned} 2e = e_{\varphi(3)} + e_{\varphi(1)} &\geqslant e_{\varphi(4)} + e_{\varphi(2)} \qquad \text{(from (14))} \\ &\geqslant 2e. \end{aligned}$$

Hence $e_{\varphi(2)} = e_{\varphi(4)} = e$. Also for any i, $4 < i \leqslant k$, we have

$$\begin{aligned} 2e = e_{\varphi(1)} + e_{\varphi(2)} &\geqslant e_{\varphi(i-1)} + e_{\varphi(i)} \\ &\geqslant 2e. \end{aligned}$$

Hence $e_{\varphi(i)} = e$, $i = 1,..., k$.

Conversely, (15) and (17) clearly imply (14)–(16) and the proof is complete.

We will use the following lemma to prove Theorem 1.4.

LEMMA 2.4. *Let $x_1 ,..., x_k$ be distinct elements of a field F of characteristic* 0, *k even, and let $A = \{x_k - x_i : i = 1,..., k - 1\}$. Then there exists an integer t_0, $1 \leqslant t_0 \leqslant k - 1$ such that $x_{t_0} - x_k \notin A$.*

Proof. Let $B = \{x_i - x_k : i = 1,..., k - 1\}$. Suppose $B \subset A$. Since two elements of B cannot be identical with the same element of A it follows that $B = A$. Hence there exists a permutation $\sigma \in S_{k-1}$ such that

$$x_k - x_i = x_{\sigma(i)} - x_k , \qquad i = 1,..., k - 1. \tag{18}$$

We observe that σ has no fixed point because if $\sigma(j) = j$, $1 \leqslant j \leqslant k - 1$, then from (18), $x_k - x_j = x_{\sigma(j)} - x_k = x_j - x_k$, in contradiction to the hypothesis. Thus, since $k - 1$ is odd, in the cycle decomposition of σ there is at least one cycle of odd length. We can assume without loss of generality that such a cycle is of the form $(1\ 2 \cdots 2p + 1)$. Then $x_k - x_1 = x_2 - x_k$ and $x_k - x_{2p+1} = x_1 - x_k$. But these relations imply that $x_2 = x_{2p+1}$, a contradiction. Hence B is not contained in A and the result follows.

Proof of Theorem 1.4. Reorder $\gamma_1 ,..., \gamma_k$ so that $e_k = E$. Then the highest degree elementary divisor of the characteristic matrix of $A \otimes I_n - I_n \otimes A$ involving the zero eigenvalue is λ^{2E-1} and among those involving the distinct eigenvalues

$$\gamma_k - \gamma_i , \qquad i = 1,..., k - 1, \tag{19}$$

are $(\lambda - (\gamma_k - \gamma_i))^{E+e_i-1}$, $i = 1,..., k - 1$ [5]. Theorem 1.3 tells us that there are at least $k - 1$ more distinct nonzero eigenvalues of the form $\gamma_s - \gamma_t$, $s \neq t$, $s \neq k$, which are distinct from those in (19). Hence

$$\begin{aligned} d_A &\geqslant (2E - 1) + \sum_{i=1}^{k-1} (E + e_i - 1) + \sum_{s,t} (e_s + e_t - 1) \\ &\geqslant 2E - 1 + (k - 1)(E - 1) + \sum_{i=1}^{k-1} e_i + (k - 1)(2e - 1) \\ &= kE + m + 2(k - 1)e - 2k + 1, \end{aligned}$$

where $\sum_{s,t}$ indicates the summation over $(k-1)$ pairs (s, t). This proves (5). To prove (6), we observe that if k is even, then by Lemma 2.4 there exists an integer t_0, $1 \leqslant t_0 \leqslant k-1$, such that $\gamma_{t_0} - \gamma_k$ is distinct from the eigenvalues in (19). Thus

$$d_A \geqslant 2E - 1 + \sum_{i=1}^{k-1} (E + e_i - 1) + (e_{t_0} + E - 1) + (k-2)(2e-1)$$
$$\geqslant (k+1)E + m + (2k-3)e - 2k + 1,$$

proving (6). Notice that if k is even then the bound in (6) is always greater than the one in (5) unless $E = e$ in which case they are equal.

Proof of Theorem 1.5. We immediately compute that

$$d_A = 2e_1 - 1 \qquad \text{if} \quad k = 1 \tag{20}$$

and

$$d_A = 4E + 2e - 3 \qquad \text{if} \quad k = 2. \tag{21}$$

For $k \geqslant 3$, since $m \geqslant E + (k-1)e$ and $e \geqslant 1$ we have from (13),

$$d_A \geqslant 4E + 2k - 5. \tag{22}$$

If $d_A = 1$, then the above relations tell us that A has only one eigenvalue γ and that the minimal polynomial of A is $f(\lambda) = \lambda - \gamma$. Hence A is a scalar matrix. If $d_A = 3$, then (22) implies that $k = 1$ or 2. If A has only one eigenvalue γ then, by (20), $f(\lambda) = (\lambda - \gamma)^2$ is the minimal polynomial of A and thus $A - \gamma I_n$ is nilpotent of index 2. In case A has two distinct eigenvalues γ_1 and γ_2, then it follows from (21) that $e_1 = e_2 = 1$ and A is similar to a diagonal matrix with two distinct eigenvalues. Finally, consider the case $d_A = 5$. If γ is the only eigenvalue of A then, as above $A - \gamma I_n$ is nilpotent of index 3. If A has two eigenvalues, then from (21) we have

$$2E + e = 4,$$

an impossibility because $E \geqslant e \geqslant 1$. If A has three distinct eigenvalues γ_1, γ_2, γ_3, then from (22), $E = 1$ and hence $e_1 = e_2 = e_3$ (see examples in Section 3).

To prove the last part of the theorem, let $U_n \in M_n(F)$ be the matrix with 1 in the positions $(i, i+1)$, $i = 1,..., n-1$ and 0 elsewhere. For $p = 4t - 1$, $t = 2, 3,...$ let $A = U_t \dotplus [1]$, where $\dotplus$ denotes the direct sum of matrices. Then the elementary divisors of $\lambda I_n - A$ are λ^t and $(\lambda - 1)$. It follows from (21) that $d_A = 4t - 1 = p$. For $p = 4t + 1$, $t = 2, 3,...$, let $A = U_t \dotplus \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Then the elementary divisors of the characteristic matrix of A are λ^t and

$(\lambda - 1)^2$ and again by (21), $d_A = 4t + 4 - 3 = p$. This completes the proof.
The following two lemmas will be used to prove Theorem 1.7.

LEMMA 2.5. *Let $\mathscr{S}$ be the set of distinct sums of the form $r_i + r_j$, $i, j = 1,..., p$, where $r_1,..., r_p$ are distinct rational numbers. Then*

(a) *There exists $\sigma \in S_p$ such that the set*

$$M = \{2r_i : i = 1,..., p\} \cup \{r_{\sigma(i)} + r_{\sigma(i+1)} : i = 1,..., p - 1\}$$

consists of distinct elements and hence $\nu(\mathscr{S}) \geqslant 2p - 1$; and

(b) *$\nu(\mathscr{S}) = 2p - 1$ if and only if there exist $\sigma \in S_p$, $a, b \in Q$,*

$$b \neq 0 \text{ such that } r_{\sigma(t)} = a + bt, \qquad t = 1,..., p. \tag{23}$$

Proof. Find a permutation $\sigma \in S_p$ such that $r_{\sigma(1)} < r_{\sigma(2)} < \cdots < r_{\sigma(p)}$. Set $r_{\sigma(i)} = s_i$, $i = 1,..., p$. Then the elements of $C = \{2s_i : i = 1,..., p\}$ are all distinct. Also the set $D = \{s_i + s_{i+1} : i = 1,..., p - 1\}$ consists of distinct elements. Moreover, $C \cap D = \varnothing$ because $2s_i < s_j + s_{j+1}$ for $i \leqslant j$ and $2s_i > s_j + s_{j+1}$ for $i > j$. This proves (a).

To prove (b) we first notice that $\nu(\mathscr{S})$ does not change if each r_i is replaced by $a + r_i$, by br_i or by $r_{\sigma(i)}$, $a \in Q$, $b \in Q$, $b \neq 0$, $\sigma \in S_p$. Hence $\nu(\mathscr{S}) = d$ if and only if the number of distinct sums involving $a + br_{\sigma(i)}$, $b \neq 0$, $i = 1,..., p$ is d. Since the number of distinct sums involving $1,..., p$ is $2p - 1$ it follows that if $r_{\sigma(t)} = a + bt$, $t = 1,..., p$ then $\nu(\mathscr{S}) = 2p - 1$. To prove the converse we use induction on p. For $p = 1$ the result is obvious. We assume that for $k \leqslant p - 1$ if the number of distinct sums involving $s_1,..., s_k$ is $2k - 1$, then $s_i - s_{i-1} = b$, $i = 2,..., k$, $b \neq 0$. We assert that the set of all distinct sums involving $s_1,..., s_{p-1}$ is precisely

$$N = \{2s_i : i = 1,..., p - 1\} \cup \{s_i + s_{i+1} : i = 1,..., p - 2\}.$$

For, if $s_i + s_j \notin N$ for some i and j, $1 \leqslant i, j \leqslant p - 1$, then, since M is the set of all distinct sums involving $s_1,..., s_p$, we have

$$s_i + s_j = 2s_p \qquad \text{or} \qquad s_i + s_j = s_{p-1} + s_p.$$

But none of these relations is possible because $s_i + s_j \leqslant s_{p-1} + s_{p-1} < s_{p-1} + s_p < 2s_p$. Thus by the induction hypothesis,

$$s_i - s_{i-1} = b, \qquad i = 1,..., p - 1.$$

The proof will be complete if we can show that $s_p - s_{p-1} = b$. Since $s_p + s_{p-2} \in M$, there are two possibilities, namely, $s_p + s_{p-2} = s_i + s_{i+1}$ for some i, $1 \leqslant i \leqslant p - 1$, or $s_p + s_{p-2} = 2s_i$, for some i, $i \neq p - 2$, $i \neq p$. If $i \leqslant p - 2$, then $s_i + s_{i+1} \leqslant s_{p-2} + s_{p-1} < s_{p-2} + s_p$. If $i = p - 1$, then $s_i + s_{i+1} = s_{p-1} + s_p > s_{p-2} + s_p$. Hence $s_p + s_{p-2} \neq s_i + s_{i+1}$ for any $i = 1,..., p - 1$. Since $s_p + s_{p-2} > 2s_i$, $i = 1,..., p - 2$, we conclude that $s_p + s_{p-2} = 2s_{p-1}$ and the proof is complete.

LEMMA 2.6. *Let $\varphi \in S_p$. Then the number of vectors in $V_2(Q)$ of the form $(i + j, \varphi(i) + \varphi(j))$, $i, j = 1,..., p$ is exactly $2p - 1$ if and only if either φ is the identity permutation e or $\varphi(t) = p - t + 1$, $t = 1,..., p$.*

Proof. Consider the following scheme (see next page).

The entries in the first column together with the last entry in each of the remaining $p - 1$ columns form a set of distinct vectors. Thus the number of distinct vectors of the form $(i + j, \varphi(i) + \varphi(j))$, $i, j = 1,..., p$ is $2p - 1$ if and only if each row of the given scheme contains only one vector. This is possible if and only if

$$\varphi(2) - \varphi(1) = \varphi(3) - \varphi(2) = \cdots$$
$$= \varphi(p - 1) - \varphi(p - 2) = \varphi(p) - \varphi(p - 1).$$

Let $\varphi(1) = k$ and $\varphi(2) = k + l$. Then

$$\varphi(i) = k + (i - 1)l, \qquad i = 1,..., p.$$

Now if $l > 0$, then $k + (p - 1)l = p$ and $k = 1$ which implies that $\varphi = e$. In case $l < 0$, it follows that $k + (p - 1)l = 1$ and $k = p$. This gives $\varphi(t) = p - t + 1$, $t = 1,..., p$. This completes the proof of the lemma.

In view of Lemmas 2.5 and 2.6, the proof of Theorem 1.7 is analogous to that of Theorem 1.3.

Proof of Theorem 1.6. We consider the operator $2L_A$ which has a matrix representation $A \otimes I_n + I_n \otimes A$. Among the elementary divisors of the characteristic matrix of $A \otimes I_n + I_n \otimes A$, involving the eigenvalue $\gamma_i + \gamma_j$, are

$$(\lambda - (\gamma_i + \gamma_j))^{e_i + e_j - 1}, \qquad i, j = 1,..., k \text{ [5]}.$$

In view of Theorem 1.7 we can find a permutation $\varphi \in S_k$ such that the set

$$\{2\gamma_i : i = 1,..., k\} \cup \{\gamma_{\varphi(i)} + \gamma_{\varphi(i+1)} : i = 1,..., k - 1\}$$

contains $2k - 1$ elements. Hence

$$\delta_A \geqslant \sum_{i=1}^{k} (2e_i - 1) + \sum_{i=1}^{k-1} (e_{\varphi(i)} + e_{\varphi(i+1)} - 1) \tag{24}$$

$$= 2m - k + (m - e_{\varphi(k)}) + (m - e_{\varphi(1)}) - k + 1$$

$$\geqslant 2(2m - E - k) + 1. \tag{25}$$

$i+1, \varphi(i)+\varphi(1)$	$i+2, \varphi(i)+\varphi(2)$	$i+3, \varphi(i)+\varphi(3)$	$\cdots$	$i+p, \varphi(i)+\varphi(p)$
$2, \varphi(1)+\varphi(1)$				
$3, \varphi(2)+\varphi(1)$	$3, \varphi(1)+\varphi(2)$			
$4, \varphi(3)+\varphi(1)$	$4, \varphi(2)+\varphi(2)$	$4, \varphi(1)+\varphi(3)$		
$5, \varphi(4)+\varphi(1)$	$5, \varphi(3)+\varphi(2)$	$5, \varphi(2)+\varphi(3)$	$\cdots$	
$\cdot$	$\cdot$	$\cdot$	$\cdots$	
$\cdot$	$\cdot$	$\cdot$	$\cdots$	
$\cdot$	$\cdot$	$\cdot$	$\cdots$	
$p, \varphi(p-1)+\varphi(1)$	$p, \varphi(p-2)+\varphi(2)$	$p, \varphi(p-3)+\varphi(3)$	$\cdots$	
$p+1, \varphi(p)+\varphi(1)$	$p+1, \varphi(p-1)+\varphi(2)$	$p+1, \varphi(p-2)+\varphi(3)$	$\cdots$	$p+1, \varphi(1)+\varphi(p)$
	$p+2, \varphi(p)+\varphi(2)$	$p+2, \varphi(p-1)+\varphi(3)$	$\cdots$	$p+2, \varphi(2)+\varphi(p)$
		$p+3, \varphi(p)+\varphi(3)$	$\cdots$	$p+3, \varphi(3)+\varphi(p)$
			$\cdots$	$p+4, \varphi(4)+\varphi(p)$
				$\cdot$
				$\cdot$
				$\cdot$
				$2p, \varphi(p)+\varphi(p)$

But also XS/XA has finite length as a right R-module. Thus,

$$K\dim(W_i/X)_R = K\dim(W_i/XS)_R < m. \quad \blacksquare$$

Before going any further we give an example which shows the necessity in the preceding results of the restrictions upon the right ideal A. This example is adapted from one due to Björk, [3, Example 1.1].

EXAMPLE 2.7. Let K be a field with a derivation $'$ such that the field $K_0 = \{k \in K \mid k' = 0\}$ satisfies $[K : K_0] = \infty$. Let $T = K[y]$ be the ring of commutative polynomials over K in an indeterminate y. Define a derivation d on T by

$$d(k_n y^n + \cdots + k_0) = (k_n' y^n + \cdots + k_0')y.$$

Let $S = T[x]$ be the ring of polynomials over T in an indeterminate x subject to $tx - xt = d(t)$ for $t \in T$. By [16, Theorem 4, pp. 164–5], S is a noetherian integral domain.

Let $A = xS + y^2S$ and $B = xS + yS$. It can be checked that S/A has a unique composition series of length 2, $S \supset B \supset A$, and that $\mathbf{I}(A)/A$ is isomorphic to the ring of commutative polynomials $K_0 + \bar{y}K$, $\bar{y}^2 = 0$. This ring is not noetherian, so $\mathbf{I}(A)$ cannot be right noetherian. Further, the ring $\bar{S} = S/B^2$ is artinian, and yet, for the same reason, $\mathbf{I}(A/B^2)$ is not right artinian.

In these examples it is not true that $SA = S$; in fact $SA = B$. This can be remedied. Let S' be the ring of 2×2 matrices over S and let

$$A' = \begin{pmatrix} S & S \\ A & A \end{pmatrix}, \qquad B' = \begin{pmatrix} S & S \\ B & B \end{pmatrix}.$$

Then $S' \supset B' \supset A'$ is a unique S'-composition series for S'/A' and $S'A' = S'$. It is easy to check that

$$\mathbf{I}(A') = \begin{pmatrix} S & S \\ A & \mathbf{I}(A) \end{pmatrix}.$$

Therefore $\mathbf{I}(A) \cong e_{22}\mathbf{I}(A')\, e_{22}$. Since $\mathbf{I}(A)$ is not noetherian, neither is $\mathbf{I}(A')$. ∎

We end this section with an investigation of the right global dimensions of S and R. In Lemma 2.1, it was shown that S is a finitely generated projective right R-module. The next lemma strengthens that result.

LEMMA 2.8. *Let S be a ring, A a right ideal such that $SA = S$, R a subring of S containing A, and V a right S-module. Then,* p.d. $V_S =$ p.d. V_R .

then $e_{k-1} = E$. It would then follow from the induction hypothesis that $e_i = E, i = 1,..., k - 1$. Suppose k is even, say $k = 2t$. Then $1 + k = 2t + 1$ implies that $2E = e_1 + e_k \leqslant e_t + e_{t+1}$ yielding $e_t = e_{t+1} = E$. Now one of the integers t and $t + 1$ is odd, call it k_1. Then there are integers t_1 and $t_1 + 1$ such that $k_1 + k = 2t_1 + 1$ which implies $e_{t_1} = e_{t_1+1} = E$. We continue this process until we get $t_q = k - 1$ and $e_{t_q} = E$. If k is odd, say $k = 2t - 1$, then $1 + k = 2t$ implies $e_t = E$. If $t = k - 1$, we are finished; otherwise find t_1 such that either $2t_1 = t + k$ or $2t_1 + 1 = t + k$. This gives $e_{t_1} = E$. We continue the process until we get $e_{t_q} = E$ and $t_q = k - 1$. This completes the proof.

Proof of Theorem 1.8. If k is 1 or 2, then it follows immediately that

$$\delta_A = 2e_1 - 1 \qquad \text{if} \quad k = 1 \tag{30}$$

and

$$\delta_A = 3(E + e - 1) \qquad \text{if} \quad k = 2. \tag{31}$$

Also, since $m \geqslant E + (k - 1)$, we have using (25)

$$\delta_A \geqslant 2E + 2k - 3 \tag{32}$$

$$\geqslant 2k - 1. \tag{33}$$

Moreover, if $k = 3$ and $E = 1$, then the number of distinct eigenvalues of L_A is 5 or 6. Hence

$$5 \leqslant \delta_A \leqslant 6 \qquad \text{if} \quad k = 3 \text{ and } E = 1. \tag{34}$$

Next consider the case $k = 3$ such that $e_{\sigma(1)} = 2$ and $e_{\sigma(2)} = e_{\sigma(3)} = 1$ for some $\sigma \in S_3$. Then the eigenvalues $\gamma_{\sigma(1)} + \gamma_{\sigma(i)}$, $i = 1, 2, 3$ of $2L_A$ are distinct and there are at least two more, say, $\gamma_{\sigma(i_1)} + \gamma_{\sigma(j_1)}$ and $\gamma_{\sigma(i_2)} + \gamma_{\sigma(j_2)}$, which are different from the preceding three. Thus

$$\begin{aligned}\delta_A &\geqslant \sum_{i=1}^{3} (e_{\sigma(1)} + e_{\sigma(i)} - 1) + \sum_{t=1}^{2} (e_{\sigma(i_t)} + e_{\sigma(j_t)} - 1) \\ &= 3 + 2 + 2 + 1 + 1 \\ &= 9.\end{aligned}$$

Since L_A has at most six distinct eigenvalues it is obvious from the above computation that $9 \leqslant \delta_A \leqslant 10$.

If $k = 3$, $E = 2$ and two of the e_i's are 2 and the third is 1, then there is a permutation $\varphi \in S_3$ such that the following eigenvalues of $2L_A$ are distinct: $2\gamma_{\varphi(1)}$, $2\gamma_{\varphi(2)}$, $2\gamma_{\varphi(3)}$, $\gamma_{\varphi(1)} + \gamma_{\varphi(2)}$ and $\gamma_{\varphi(2)} + \gamma_{\varphi(3)}$. If $2L_A$ has exactly five distinct eigenvalues, then $\gamma_{\varphi(1)} + \gamma_{\varphi(3)} = 2\gamma_{\varphi(2)}$. If $e_{\varphi(2)}$ is 1, then the highest degree elementary divisor of the characteristic matrix of $A \otimes I_n + I_n \otimes A$ involving $2\gamma_{\varphi(2)}$ is

$$(\lambda - 2\gamma_{\varphi(2)})^{e_{\varphi(1)}+e_{\varphi(3)}-1} = (\lambda - 2\gamma_{\varphi(2)})^3$$

and

$$\begin{aligned}\delta_A &= \sum_{i=1}^{3} (2e_{\varphi(i)} - 1) + \sum_{i=1}^{2} (e_{\varphi(i)} + e_{\varphi(i+1)} - 1) \\ &= 3 + 3 + 3 + 2 + 2 \\ &= 13,\end{aligned}$$

while if $e_{\varphi(2)} = 2$, then either $e_{\varphi(1)} = 1$ and $e_{\varphi(3)} = 2$ or $e_{\varphi(1)} = 2$ and $e_{\varphi(3)} = 1$ and in both the cases δ_A is 12.

If $k = 3$ and $e = E = 2$, then it is immediate that $\delta_A \geqslant 15$. Thus for $k = 3$ and $E = 2$ we have the following possibilities:

$$10 \geqslant \delta_A \geqslant 9 \quad \text{if} \quad e_{\sigma(1)} = 2 \quad \text{and} \quad e_{\sigma(2)} = e_{\sigma(3)} = 1, \tag{35}$$

for some $\sigma \in S_3$;

$$\delta_A \geqslant 12 \quad \text{if} \quad e_{\sigma(1)} = e_{\sigma(2)} = 2 \quad \text{and} \quad e_{\sigma(3)} = 1, \tag{36}$$

for some $\sigma \in S_3$;

$$\delta_A \geqslant 15 \quad \text{if} \quad e_i = 2, \quad i = 1, 2, 3. \tag{37}$$

Now (30) and (31) tell us that δ_A is odd if k is 1, and a multiple of 3 if k is 2. Also, from (33), we have $\delta_A \geqslant 5$ if $k \geqslant 3$. This proves (i).

To prove (ii) we consider all the cases separately.

Case 1. $\delta_A = 1$: It is easily seen that A has only one eigenvalue γ and that $\lambda - \gamma$ is the minimal polynomial of A. Hence A is a scalar matrix.

Case 2. $\delta_A = 3$: It follows from (30) and (31) that either $k = 1$ and $e_1 = 2$ or $k = 2$ and $E = 1$, i.e., A satisfies either (a) or (b).

Case 3. $\delta_A = 5$: It is obvious from (31) and (33) that $k \neq 2$ and $k < 4$. Using (30) and (32), we conclude that either $k = 1$ and $e_1 = 3$, or $k = 3$ and $E = 1$.

Case 4. $\delta_A = 7$: We observe, as in the preceding case, that $k \neq 2$ and $k < 5$. If $k = 1$, then $e_1 = 4$ and A satisfies (b). If $k = 3$ then, from (32), $E \leqslant 2$. Now (34) implies that $E \neq 1$ and (35), (36), and (37) imply that $E \neq 2$. Thus A cannot have three distinct eigenvalues when $\delta_A = 7$. If $k = 4$, then it is evident from (32) that $E = 1$ and hence A satisfies (a).

Case 5. $\delta_A = 11$: It follows from (31) and (33) that $k \neq 2$ and $k \leqslant 6$. If $k = 1$, then (30) gives $e_1 = 6$, and $A - \gamma_1 I_n$ is nilpotent of index 6.

If $k = 3$ then it follows from (34)–(37) that $E \neq 1$ or 2. In case $E = 3$, then it is immediate from (25) that at most one e_i is 3 and each of the remaining two is 1. So assume $e_{\sigma(1)} = 3$, $e_{\sigma(2)} = e_{\sigma(3)} = 1$ for some $\sigma \in S_3$. Now the eigenvalues $2\gamma_{\sigma(1)}$, $\gamma_{\sigma(1)} + \gamma_{\sigma(2)}$, and $\gamma_{\sigma(1)} + \gamma_{\sigma(3)}$ of $2L_A$ are distinct and hence

$$\delta_A > (2e_{\sigma(1)} - 1) + (e_{\sigma(1)} + e_{\sigma(2)} - 1) + (e_{\sigma(1)} + e_{\sigma(3)} - 1) = 11.$$

A similar argument shows that $E \neq 4$. To analyze the case $k = 4$ we first notice that if each e_i is 1, then, since the number of distinct eigenvalues of L_A in this case is at most 10, $\delta_A \leqslant 10$. If one of the e_i's, say, e_1, is 2, and $e_2 = e_3 = e_4 = 1$, then the eigenvalues $\gamma_1 + \gamma_i$, $i = 1, 2, 3, 4$ of $2L_A$ are distinct and there are at least three more. Hence $\delta_A \geqslant \sum_{i=1}^{4} (e_1 + e_i - 1) + 3 = 12$. Thus $E \neq 2$. Similarly, $E \neq 3$. Since, from (32), $E \leqslant 3$ we have proved that if $\delta_A = 11$, then A cannot have four distinct eigenvalues. If $k = 5$, then $E \leqslant 2$. But as in the case $k = 4$, it is easily seen that $E \neq 2$ and hence $E = 1$. Finally, if $k = 6$ then we use (32) to conclude that $E = 1$. Thus A satisfies either (a) or (b). Examples to illustrate these cases are given in Section 3.

(iii) If $\delta_A = 8$, then it is immediate from (33), (30) and (31) that $k < 5$, $k \neq 1$ and $k \neq 2$. If $k = 3$, then (32) implies that $E \leqslant 2$ but this is not possible in view of (34)–(37). Thus the only possibility for k is 4 and then, from (32), $E = 1$.

(iv) (a) For $p = 4t + 1$, $t = 2, 3, 4, \ldots$, consider the matrix

$$A = (I_t + U_t) \dotplus \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \in M_{t+2}(F).$$

Then the highest degree elementary divisors of $\lambda I_n - A$ are $(\lambda - 1)^t$, $(\lambda - 2)$, and $(\lambda - 3)$ and hence those of the characteristic matrix of $A \otimes I_n + I_n \otimes A$, $n = t + 2$, are

$$(\lambda - 2)^{2t-1}, \qquad (\lambda - 3)^t, \qquad (\lambda - 4)^t, \qquad (\lambda - 5) \qquad \text{and} \qquad (\lambda - 6).$$

Thus $\delta_A = 4t + 1 = p$.

(b) For $p = 4t + 2$, $t = 2, 3, 4, \ldots$, let

$$A = (I_t + U_t) \dotplus \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \in M_n(F), \qquad n = t + 2.$$

Then the highest degree elementary divisors of $\lambda I_n - A$ are $(\lambda - 1)^t$, $(\lambda - 2)$, and $(\lambda - 4)$ and hence those of the characteristic matrix of $A \otimes I_n + I_n \otimes A$

are $(\lambda - 2)^{2t-1}$, $(\lambda - 3)^t$, $(\lambda - 4)$, $(\lambda - 5)^t$, $(\lambda - 6)$ and $(\lambda - 8)$ yielding $\delta_A = 4t + 2 = p$.

For $p = 6$, set

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{so that} \quad \delta_A = 6.$$

(c) If $p = 4t + 3$, $t \geqslant 3$, then consider

$$A = (I_t + U_t) \dotplus \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \in M_n(F), \qquad n = t + 3.$$

Then, as before, the highest degree elementary divisors of the characteristic matrix of $A \otimes I_n + I_n \otimes A$ are $(\lambda - 2)^{2t-1}$, $(\lambda - 3)^{t+1}$, $(\lambda - 4)^t$, $(\lambda - 5)^2$, and $(\lambda - 6)$ and hence $\delta_A = 4t + 3 = p$.

(d) If $p = 4t + 4$, $t \geqslant 2$, then set

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \dotplus (2I_t + U_t) \in M_n(F), \qquad n = t + 3$$

Then the highest degree elementary divisors of $\lambda I_n - A$ are $(\lambda - 1)^2$, $(\lambda - 2)^t$, and $(\lambda - 3)$, and hence those of the characteristic matrix of $A \otimes I_n + I_n \otimes A$ are $(\lambda - 2)^3$, $(\lambda - 3)^{t+1}$, $(\lambda - 4)^{2t-1}$, $(\lambda - 5)^t$ and $(\lambda - 6)$, so that $\delta_A = 4t + 4 = p$.

Notice that the matrix A in (a), (b), (c), or (d) is not similar to a diagonal matrix, nor is $(A - \alpha I_n)$ nilpotent for any $\alpha \in K$. This completes the proof of Theorem 1.8.

3. Examples

The following two examples show that the two lower bounds in (1) and (5) are not comparable. Denote the right sides of (1) and (5) by d_1 and d_2, respectively.

(1) $d_1 > d_2$: Let A be the matrix with three distinct eigenvalues γ_1, γ_2, and γ_3 such that $e_1 = e_2 = 2$ and $e_3 = 1$. Then $d_1 = 11$ and $d_2 = 10$.

(2) $d_1 < d_2$: Let $k = 4$, $e_1 = 2$, and $e_2 = e_3 = e_4 = 1$. Then $d_1 = 11$, $d_2 = 12$.

The following three examples illustrate the first part of Theorem 1.5.

(3) A scalar matrix A satisfies both of the conditions (i) and (ii) of the theorem and $d_A = 1$.

(4) A diagonal matrix A with two distinct eigenvalues satisfies (i) and $d_A = 3$, while for $n \geqslant 3$ if $A = E_{1n}$ (E_{1n} is the matrix with 1 in position $(1, n)$ and 0 elsewhere) then A is nilpotent of index 2 and $d_A = 3$.

(5) A diagonal matrix with distinct eigenvalues 1, 2 and 3 has $d_A = 5$, while if $A = U_3$, then $d_A = 5$ and A is nilpotent of index 3.

The following examples illustrate parts (ii) and (iii) of Theorem 1.8.

(6) Let $A \in M_n(F)$ be a diagonal matrix with distinct eigenvalues 1, 3, 4, and 5. Then the distinct eigenvalues of $A \otimes I_n + I_n \otimes A$ are 2, 4, 5, 6, 7, 8, 9, and 10, respectively, and all the elementary divisors of its characteristic matrix are linear. Hence $\delta_A = 8$.

(7) If $A \in M_n(F)$ is a diagonal matrix and the distinct eigenvalues of A are $1, \ldots, p$, where p is 1, 2, 3, 4 or 6, then δ_A is 1, 3, 5, 7 or 11, respectively, and if the distinct eigenvalues of A are 1, 4, 5, 6 and 7, then $\delta_A = 11$.

(8) A scalar matrix is nilpotent of index 1 and $\delta_A = 1$. For $p = 3$, 5, 7 and 11, consider

$$A = U_t, \qquad t = \frac{p+1}{2}.$$

Then A is nilpotent of index $(p + 1)/2$ and $\delta_A = p$.

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